

A Study of Continued Fractions Using Software Tools – I

Cătălin Angelo IOAN¹, Alin Cristian IOAN²

Abstract. The paper treats some properties of continuous fractions using appropriate software.

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1 Introduction

We call continued fractions a number which has the expression:

$$\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ddots + \frac{1}{a_n + \ddots}}}$$

where $a_0 \in \mathbb{Z}$, $a_i \in \mathbb{N}^* \forall i \geq 1$.

We shall write: $\alpha = [a_0, a_1, \dots, a_n, \dots]$.

With Euclid's algorithm every rational number has as a finite representation like a continued fraction. Because, for $\alpha \in \mathbb{Q}$, if $a_n \geq 2$: $\alpha = [a_0, a_1, \dots, a_n] = [a_0, a_1, \dots, a_n - 1, 1]$ we shall consider the shorter decomposition for α therefore the first expression.

We shall remind, in what follows, from the [2], the principal properties of continuous fractions.

Let note $\alpha_n = [a_0, a_1, \dots, a_n]$ - the convergent of α of order n and $\alpha_n = \frac{p_n}{q_n}$, $(p_n, q_n) = 1$. p_n and q_n are

called the continuants of α of order n .

It is easily to see that:

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2} \\ q_n = a_n q_{n-1} + q_{n-2} \end{cases}, n \geq 1$$

where: $p_{-1} = 1$, $p_0 = a_0$, $q_{-1} = 0$, $q_0 = 1$.

After these definitions and notations, we have:

$$\alpha_n = \frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} = \frac{p_{n-1}}{q_{n-1}} + \frac{\frac{p_{n-2}}{q_{n-1}} - \frac{p_{n-1}}{q_{n-2}}}{a_n \frac{q_{n-1}}{q_{n-2}} + 1} = \alpha_{n-1} - \frac{\alpha_{n-1} - \alpha_{n-2}}{a_n \frac{q_{n-1}}{q_{n-2}} + 1}$$

therefore:

$$\alpha_n - \alpha_{n-1} = -\frac{\alpha_{n-1} - \alpha_{n-2}}{a_n \frac{q_{n-1}}{q_{n-2}} + 1} = -\frac{q_{n-2}}{a_n q_{n-1} + q_{n-2}} (\alpha_{n-1} - \alpha_{n-2}) = -\frac{q_{n-2}}{q_n} (\alpha_{n-1} - \alpha_{n-2})$$

¹ Danubius University of Galati, Department of Finance and Business Administration, catalin_angelo_ioan@univ-danubius.ro

² Nicolae Onicescu College, Braila, alincristianioan@yahoo.com

from where:

$$\alpha_n - \alpha_{n-1} = -\frac{q_{n-2}}{q_n}(\alpha_{n-1} - \alpha_{n-2}) = \frac{q_{n-2}}{q_n} \frac{q_{n-3}}{q_{n-1}}(\alpha_{n-2} - \alpha_{n-3}) = \dots = (-1)^{n-1} \frac{q_{n-2}}{q_n} \frac{q_{n-3}}{q_{n-1}} \dots \frac{q_0}{q_2}(\alpha_1 - \alpha_0) =$$

$$(-1)^{n-1} \frac{q_0 q_1}{q_{n-1} q_n}(\alpha_1 - \alpha_0) = (-1)^{n-1} \frac{a_1}{q_{n-1} q_n} \left(a_0 + \frac{1}{a_1} - a_0 \right) = \frac{(-1)^{n-1}}{q_{n-1} q_n}$$

It is easily to see that:

- $q_n p_{n-1} - q_{n-1} p_n = (-1)^n$
- $(p_n, q_n) = 1$
- $\alpha_n - \alpha_{n-1} = \frac{(-1)^{n-1}}{q_{n-1} q_n}$
- $|\alpha_p - \alpha_n| < |\alpha_s - \alpha_n| \quad \forall s < p < n$
- $(\alpha_{2k})_{k \geq 1, 2k < n}$ is increasing, but $\alpha_{2k} < \alpha_n$
- $(\alpha_{2k+1})_{k \geq 0, 2k+1 < n}$ is decreasing, but $\alpha_{2k+1} > \alpha_n$
- $\frac{1}{q_n(q_n + q_{n+1})} < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}$

After the last statement we can see that any convergent is closer to the value α of the continued fraction than any other rational whose denominator is less than that of the convergent.

Also, the theorem of Lagrange states that any quadratic irrational (that is a number which is root of an equation of second degree with discriminant not perfect square) has a periodic development as continued fraction, that is:

$$a + b\sqrt{c} = [a_0, a_1, \dots, \overline{a_k, \dots, a_n}]$$

which means that $a_{n+1} = a_k$, $a_{n+2} = a_{k+1}$ and so on.

The reciprocal is also true.

2 Solving algebraic equations with the aid of continuous fractions

Let now the polynomial $P \in \mathbb{Z}[X]$ and the equation $P=0$. First, we isolate the roots in intervals with whole ends. Let, for example: $x_1 \in (a_0, a_0+1)$, $a_0 \in \mathbb{Z}$. It is not necessary to have only one root in this interval.

The associate Taylor polynomial is:

$$P(a_0 + \alpha) = P(a_0) + \frac{P'(a_0)}{1!} \alpha + \frac{P''(a_0)}{2!} \alpha^2 + \dots + \frac{P^{(n)}(a_0)}{n!} \alpha^n \quad \forall \alpha \in \mathbb{R}$$

Therefore:

$$P\left(a_0 + \frac{1}{\alpha}\right) = \frac{1}{\alpha^n} \left(P(a_0) \alpha^n + \frac{P'(a_0)}{1!} \alpha^{n-1} + \frac{P''(a_0)}{2!} \alpha^{n-2} + \dots + \frac{P^{(n)}(a_0)}{n!} \right)$$

If we find $\alpha \geq 1$ such that: $P\left(a_0 + \frac{1}{\alpha}\right) P\left(a_0 + \frac{1}{\alpha+1}\right) < 0$ that is:

$$\left(P(a_0) \alpha^n + \frac{P'(a_0)}{1!} \alpha^{n-1} + \frac{P''(a_0)}{2!} \alpha^{n-2} + \dots + \frac{P^{(n)}(a_0)}{n!} \right) \cdot$$

$$\left(P(a_0) (\alpha+1)^n + \frac{P'(a_0)}{1!} (\alpha+1)^{n-1} + \frac{P''(a_0)}{2!} (\alpha+1)^{n-2} + \dots + \frac{P^{(n)}(a_0)}{n!} \right) < 0$$

the root x_1 belongs in the interval: $\left(a_0 + \frac{1}{\alpha + 1}, a_0 + \frac{1}{\alpha}\right)$.

Using the Horner's diagram, the computations are organized as follows, for $P=c_nX^n+\dots+c_0$

	c_n	c_{n-1}	\dots	c_2	c_1	c_0
a_0	c_n	b_{n-1}	\dots	b_2	b_1	$P(a_0)$
	\dots	\dots	\dots	\dots	$\frac{P'(a_0)}{1!}$	
	\dots	\dots	\dots	$\frac{P''(a_0)}{2!}$		
	\dots	\dots	\dots			
	$\frac{P^{(n)}(a_0)}{n!}$					

The new diagram has new coefficients:

	$P(a_0)$	$\frac{P'(a_0)}{1!}$	$\frac{P''(a_0)}{2!}$	\dots	$\frac{P^{(n)}(a_0)}{n!}$
α					

If at a step we find two or more α such that $P\left(a_0 + \frac{1}{\alpha}\right)P\left(a_0 + \frac{1}{\alpha + 1}\right) < 0$ it follows that in the interval $(a_0, a_0 + 1)$ belongs more than one roots of the ploynomial.

3 Numbers expressed like ratio between prime numbers

All over in this paper, the software presented was written in Wolfram Mathematica 9.0.

Let now a number $r \in \mathbb{Q}_+^*$, $r = \frac{p_k}{p_s}$ where p_k – is the k-th prime number.

First, we will inquire into the existence of a development in continued fraction where all terms are primes or 1.

The next software find the development with maxim length for first 2000 prime numbers.

```
Clear["Global`*"];
numberprimes=2000;
nr=0;
For[k=2,k<=numberprimes,k++,For[s=1,s<=k-1,s++,x=Prime[k]/Prime[s];
a=ContinuedFraction[x];b=1;For[p=1,p<=Length[a],p++,x=Part[a,p];If[PrimeQ[x]||x==
1,b=b,b=0*b]];If[b==1,nr=nr+1;If[nr==1,c=a;d=k;e=s,If[Length[a]>Length[c],c=a;d=k;e=s]]]]
Print[d,"---",e,"---",Prime[d],"/",Prime[e],"=",c]
```

The result of the execution is:

$$1430\text{---}1158\text{---}11933/9349=\{1,3,1,1,1,1,1,1,1,1,1,1,1,1,2\}$$

which means that $\frac{p_{1430}}{p_{1158}} = \frac{11933}{9349} = [1,3,1,1,1,1,1,1,1,1,1,1,1,1,2]$.

If we shall find the development in continued fraction where all terms are primes we find:

```
Clear["Global`*"];
```

```

numberprimes=2000;
nr=0;
For[k=2,k≤numberprimes,k++,For[s=1,s≤k-1,s++,x=Prime[k]/Prime[s];
a=ContinuedFraction[x];b=1;For[p=1,p≤Length[a],p++,x=Part[a,p];If[PrimeQ[x],b=b,b=0*b]];
If[b==1,nr=nr+1;If[nr==1,c=a;d=k;e=s,If[Length[a]>Length[c],c=a;d=k;e=s]]]]
Print[d,"---",e,"---",Prime[d],"/",Prime[e],"=",c]

```

and the result of the execution is:

$$1271\text{---}613\text{---}10357/4517=\{2,3,2,2,2,2,2,3,2\}$$

which means that $\frac{p_{1271}}{p_{613}} = \frac{10357}{4517} = [2,3,2,2,2,2,2,3,2]$.

If we shall find the development in continued fraction where all terms are different primes we find, for first 5000 prime numbers:

```

Clear["Global`*"];
numberprimes=5000;
nr=0;
For[k=2,k≤numberprimes,k++,
For[s=1,s≤k-1,s++,x=Prime[k]/Prime[s];
a=ContinuedFraction[x];b=1;
For[p=1,p≤Length[a]-1,p++,
x=Part[a,p];
If[PrimeQ[x],
For[i=p+1,i≤Length[a],i++,
y=Part[a,i];
If[PrimeQ[y],
If[x==y,Goto[salt]],Goto[salt]]],Goto[salt]]];
If[b==1,nr=nr+1;If[nr==1,c=a;d=k;e=s,If[Length[a]>Length[c],c=a;d=k;e=s]]];
Print[k,"---",s,"---",Prime[k],"/",Prime[s],"=",a];
Label[salt]]
Print["Maximum length is for: ",d,"---",e,"---",Prime[d],"/",Prime[e],"=",c]

```

and the result of the execution is:

$$\text{Maximum length is for: } 3636\text{---}1335\text{---}33961/10993=\{3,11,5,7,13,2\}$$

which means that $\frac{p_{3636}}{p_{1335}} = \frac{33961}{10993} = [3,11,5,7,13,2]$.

In what follows we shall try to find the numbers $r \in \mathbf{Q}_+^*$, $r = \frac{p_k}{p_s}$ where p_k – is the k -th prime

number, $r = [a_0, a_1, \dots, a_n]$ for which the reverse: $s = [a_n, a_{n-1}, \dots, a_0]$ is also of the form: $s = \frac{p_u}{p_v}$.

```

Clear["Global`*"];
numberprimes=2000;
nr=0;
For[k=2,k≤numberprimes,k++,
For[s=1,s≤k-1,s++,x=Prime[k]/Prime[s];
a=ContinuedFraction[x];
lung= Length[a];
c=a;
For[p=1,p≤lung,p++,
c[[p]]=a[[lung+1-p]]];

```

```

value=FromContinuedFraction[c];
If[PrimeQ[Numerator[value]]&&PrimeQ[Denominator[value]],nr=nr+1;If[nr==1,t=a;d=k;e=s,
If[Length[a]>Length[t],t=a;u=c;d=k;e=s]]]]
Print[d,"---",e,"---",Prime[d],"/",Prime[e],"=",t," and reverse:",
Numerator[FromContinuedFraction[u]],"/",Denominator[FromContinuedFraction[u]], "=",u]

```

We found the maximum length for the first 2000 prime numbers:

$$1400---871---11657/6763=\{1,1,2,1,1,1,1,1,1,1,3,1,1,1,1,2\}$$

$$\text{and reverse: } 11657/4447=\{2,1,1,1,1,1,3,1,1,1,1,1,1,1,2,1,1\}$$

that is: $\frac{p_{1400}}{p_{871}} = \frac{11657}{6763} = [1,1,2,1,1,1,1,1,1,1,3,1,1,1,1,2]$ and the reverse development is: $\frac{11657}{4447} = [2,1,1,1,1,1,3,1,1,1,1,1,1,1,2,1,1]$, both 11657 and 4447 being primes.

4 References

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